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# Some problems in plane thermopiezoelectric materials with holes

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**Abstract**

Piezoelectric materials have recently attracted considerable attention due to their potential use in intelligent structural systems. In this paper, we treat the plane problem of thermopiezoelasticity with various holes and subject to coupled mechanical, electric and thermal loads. An analytical solution is obtained by applying the technique of conformal mapping and some identities in the Stroh formalism. The solution has a simple unified form for various holes such as ellipse, circle, triangle and square. By way of the solution, the expressions for the energy release rate and stress intensity factors of cracks are presented. Numerical results for concentration coefficients of stress and electric displacement along the hole boundary are given to assess the acceptability of the proposed method. 

## 1. Introduction

There has been considerable work done on the problem of determining the state of stress in an elastic solid containing a two-dimensional (2-D) cavity under the condition of plane strain or plane stress. Among the methods, Airy stress functions (Greenspan, 1944) and the complex variable method (Jasiuk et al., 1994) are often used. Evan-Iwanoski (1956) used the complex variable approach to derive the stress solutions for an infinite isotropic plate with a triangular inlay. Zimmerman (1986) studied the compressibility of holes by way of conformal mapping of a hole onto a unit circle. Kachanov et al. (1994) developed a unified description concerning both cavities and cracks. For orthotropic plates with rectangular openings, work has been done by Jong (1981) and Rajaiah and Naik (1983). Their results were based on the solutions given by Lekhnitskii (1968), which are only approximate solutions due to the mathematical difficulties involved. In the literature, however, there are very few works dealing with thermal stresses disturbed by holes in

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an elastic material. Florence and Goodier (1960) studied the thermal stress for an isotropic medium containing an insulated oval hole. Based on the complex variable method, Chen (1967) studied the case of an orthotropic medium with a circular or elliptic hole, and obtained a complex form solution for the hoop stress around the hole. For plane piezoelectric material without considering thermal effect, Pak (1990) and Sosa (1991, 1992) analysed some of the characteristics governing the electromechanical phenomena that arise in piezoelectric media containing defects. Zhang and Tong (1996) studied the fracture problem for a mode III crack in a piezoelectric material. Recently, Hwu (1990b) obtained the thermal stress for an anisotropic elastic plate with an elliptic hole subjected to remote uniform heat flow in the $x_2$-direction. His analysis was based on the Stroh formalism and conformal mapping.

The purpose of this paper is to present a unified description for a plane thermopiezoelectric sheet with a hole of various shapes and subjected to mixed mechanical, electric and thermal loads. Based on the extended Stroh formalism and conformal mapping, an explicit solution for the hole problem is obtained through an appropriate assumption of the form of an arbitrary function $\mathbf{F}(\mathbf{Z})$ and utilising some known identities (Ting, 1988), which enable us to convert the complex form solution into a real form. Using the solution, the expressions for the energy release rate and stress intensity factors of cracks are obtained. Numerical results for concentration coefficients of stress and electric displacement along the hole boundary are presented and comparison is made with those obtained from finite element method.

## 2. Basic equations and expressions

In this section, we shall recall briefly the governing field equations and some expressions of plane thermopiezoelectric medium. For a complete derivation and discussion the reader may refer to Wu (1984) and Yu and Qin (1996). Consider a 2-D thermoelectroelastic problem, where all field quantities are functions of $x_1$ and $x_2$ only. For convenience, shorthand notations introduced by Barnett and Lothe (1975) are adopted in the paper. In the stationary case when no free electric charge, body force and heat source are assumed to exist, the complete set of governing equations for uncoupled thermo-electroelastic problems are (Mindlin, 1974)

$$h_{i,i} = 0$$
$$\Pi_{iJ,i} = 0 \tag{1}$$

together with

$$h_i = -k_{ij}T_{,j}$$
$$\Pi_{iJ} = E_{iJKm}U_{K,m} - \chi_{iJ}T \tag{2}$$

in which

$$\Pi_{iJ} = \begin{cases} \sigma_{ij} & i, J = 1, 2, 3 \\ D_i & J = 4; \quad i = 1, 2, 3 \end{cases}$$

$$U_J = \begin{cases} u_j & J = 1,2,3 \\ \vartheta & J = 4 \end{cases} \tag{3}$$

$$\chi_{iJ} = \begin{cases} \gamma_{ij} & i, J = 1,2,3 \\ g_i & J = 4; \quad i = 1,2,3 \end{cases}$$

$$E_{iJKm} = \begin{cases} C_{ijkm} & i, J, K, m = 1,2,3 \\ e_{mij} & K = 4; \quad i, J, m = 1,2,3 \\ e_{ikm} & J = 4; \quad i, K, m = 1,2,3 \\ -\kappa_{im} & J = K = 4; \quad i, m = 1,2,3 \end{cases} \tag{4}$$

where $T$ and $h_i$ are temperature change and heat flux, $u_i$, $\vartheta$, $\sigma_{ij}$ and $D_i$ are elastic displacement, electric potential, stress and electric displacement, $C_{ijkm}$, $e_{ijk}$ and $\kappa_{ij}$ are elastic moduli, piezoelectric and dielectric constants, and $k_{ij}$, $\gamma_{ij}$ and $g_i$ are the coefficients of heat conduction, thermal-stress constants and pyroelectric constants, respectively. A general solution to (1) can be expressed as (Wu, 1984; Yu and Qin, 1996)

$$T = 2\,\mathrm{Re}\,\{g'(z_t)\}$$
$$\mathbf{U} = 2\,\mathrm{Re}\,[\mathbf{AF}(\mathbf{Z})\mathbf{q} + \mathbf{c}g(z_t)] \tag{5}$$

with

$$\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3 \quad \mathbf{A}_4]$$
$$\mathbf{F}(\mathbf{Z}) = \mathrm{diag}\,[F(z_1) \quad F(z_2) \quad F(z_3) \quad F(z_4)]$$
$$q = \{q_1 \quad q_2 \quad q_3 \quad q_4\}^T$$
$$z_t = x_1 + \tau x_2$$
$$z_i = x_1 + p_i x_2$$

in which 'Re' stands for the real part, the prime (′) denotes differentiation with the argument, $g$ and $\mathbf{F}$ are arbitrary functions to be determined, $q_i$ are complex constants determined by the related boundary conditions, $p_i$, $\tau$, $\mathbf{A}$ and $\mathbf{C}$ are constants determined by

$$k_{22}\tau^2 + (k_{12} + k_{21})\tau + k_{11} = 0$$
$$[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)p_i + \mathbf{T}p_i^2]\mathbf{A}_i = 0$$
$$[\mathbf{Q} + (\mathbf{R} + \mathbf{R}^T)\tau + \mathbf{T}\tau^2]\mathbf{c} = \chi_1 + \tau\chi_2 \tag{6}$$

in which superscript '$T$' denotes the transpose, $\boldsymbol{\chi}_i$ are $4 \times 1$ vectors, and $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are $4 \times 4$ matrices defined by

$$\boldsymbol{\chi}_i = \{\gamma_{i1} \quad \gamma_{i2} \quad \gamma_{i3} \quad g_i\}^T, \quad (\mathbf{Q})_{IK} = E_{1IK1}, \quad (\mathbf{R})_{IK} = E_{1IK2}, \quad (\mathbf{T})_{IK} = E_{2IK2} \tag{7}$$

The heat flux $\mathbf{h}$ and the stress-electric displacement (SED) $\Pi$ obtained from (2) can be written as

$$h_i = -2\,\mathrm{Re}\,\{(k_{i1}+\tau k_{i2})g''(z_t)\}$$

$$\Pi_{1J} = -\boldsymbol{\phi}_{J,2}, \quad \Pi_{2J} = \boldsymbol{\phi}_{J,1} \tag{8}$$

where $\boldsymbol{\phi}$ is the SED function given as

$$\boldsymbol{\phi} = 2\,\mathrm{Re}\,\{\mathbf{BF}(\mathbf{Z})\mathbf{q}+\mathbf{d}g(z_t)\} \tag{9}$$

with

$$\mathbf{B} = \mathbf{R}^T\mathbf{A}+\mathbf{TAP} = -(\mathbf{QA}+\mathbf{RAP})\mathbf{P}^{-1}$$

$$\mathbf{P} = \mathrm{diag}\,[p_1 \quad p_2 \quad p_3 \quad p_4]$$

$$\mathbf{d} = (\mathbf{R}^T+\tau\mathbf{T})\mathbf{c}-\boldsymbol{\chi}_2 = -(\mathbf{Q}+\tau\mathbf{R})\mathbf{c}/\tau+\boldsymbol{\chi}_1/\tau \tag{10}$$

Further, some identities are introduced in order to make the ensuing derivation tractable. Following Barnett and Lothe (1975), Ting (1988), and Chung and Ting (1995), it can be shown that the following identities are valid:

$$2\mathbf{AP}(\omega)\mathbf{A}^T = \mathbf{N}_2(\omega)-i[\mathbf{N}_2(\omega)\mathbf{S}^T+\mathbf{N}_1(\omega)\mathbf{H}]$$

$$2\mathbf{AP}(\omega)\mathbf{B}^T = \mathbf{N}_1(\omega)-i[\mathbf{N}_1(\omega)\mathbf{S}-\mathbf{N}_2(\omega)\mathbf{L}] \tag{11}$$

$$2\mathbf{BP}(\omega)\mathbf{A}^T = \mathbf{N}_1^T(\omega)-i[\mathbf{N}_1^T(\omega)\mathbf{S}^T+\mathbf{N}_3(\omega)\mathbf{H}]$$

$$2\mathbf{BP}(\omega)\mathbf{B}^T = \mathbf{N}_3(\omega)-i[\mathbf{N}_3(\omega)\mathbf{S}-\mathbf{N}_1^T(\omega)\mathbf{L}] \tag{12}$$

where $\mathbf{P}(\omega)$ is a diagonal matrix defined by

$$\mathbf{P}(\omega) = \mathrm{diag}\,[p_1(\omega) \quad p_2(\omega) \quad p_3(\omega) \quad p_4(\omega)]$$

in which $\omega$ is a rotation angle, and

$$p_i(\omega) = \frac{p_i\cos\omega-\sin\omega}{p_i\sin\omega+\cos\omega}$$

$$\mathbf{N}_1(\omega) = -\mathbf{T}^{-1}(\omega)\mathbf{R}^T(\omega), \quad \mathbf{N}_2(\omega) = \mathbf{T}^{-1}(\omega)$$

$$\mathbf{N}_3(\omega) = \mathbf{R}(\omega)\mathbf{T}^{-1}(\omega)\mathbf{R}^T(\omega)-\mathbf{Q}(\omega)$$

$$Q_{JK}(\omega) = n_i(\omega)E_{iJKm}n_m(\omega), \quad R_{JK}(\omega) = n_i(\omega)E_{iJKm}m_m(\omega)$$

$$T_{JK}(\omega) = m_i(\omega)E_{iJKm}m_m(\omega)$$

$$\mathbf{n} = \{\cos\omega \quad \sin\omega \quad 0\}^T, \quad \mathbf{m} = \{-\sin\omega \quad \cos\omega \quad 0\}^T$$

while **S**, **H**, **L** are $4\times 4$ real matrices introduced by Barnett and Lothe (1975):

$$\mathbf{S} = i(2\mathbf{AB}^T-\mathbf{I}) = \frac{1}{\pi}\int_0^{\pi}\mathbf{N}_1(\omega)\,\mathrm{d}\omega$$

$$\mathbf{H} = 2i\mathbf{AA}^T = \frac{1}{\pi}\int_0^{\pi}\mathbf{N}_2(\omega)\,\mathrm{d}\omega$$

$$\mathbf{L} = -2i\mathbf{B}\mathbf{B}^T = \frac{-1}{\pi}\int_0^{\pi} \mathbf{N}_3(\omega)\,\mathrm{d}\omega \tag{13}$$

where $\mathbf{I}$ is the unit tensor, $i = \sqrt{-1}$, $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite, and $\mathbf{SH}$, $\mathbf{LS}$, $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{SL}^{-1}$ are anti-symmetric.

## 3. Solution to the hole problem

### 3.1. Boundary conditions

Consider a plane thermoelectroelastic problem of a single insulated hole in a piezoelectric sheet which is subjected to uniform remote heat flow $\mathbf{h}^0$ and SED $\Pi^0$. The hole boundary is assumed to be traction-charge free with zero heat flow. Additionally, the hole can be thought of as being filled with air, which has a dielectric constant approximately three orders of magnitude smaller than the dielectric constant of the piezoelectric material. The consequence of such an assumption is that the boundary conditions on the hole boundary are given by $\Pi \cdot \mathbf{m} = 0$, where $\mathbf{m}$ is outward normal to the hole boundary. This is also equivalent to setting $\mathbf{E}^{\mathrm{h}} = 0$, where $\mathbf{E}^{\mathrm{h}}$ stands for the material constants of the hole-phase. Discussions on the validity of the electrical boundary conditions can be found in the literature (Dunn, 1994; Parton and Kudryatvsev, 1988).

To study the effect of the hole on the thermo-electroelastic field, it suffices to consider the associated problem in which the hole surface satisfies the conditions

$$h_m = h_1^0 \sin\theta - h_2^0 \cos\theta$$
$$\mathbf{t}_m = -\Pi_2^0 \cos\theta + \Pi_1^0 \sin\theta \tag{14}$$

along the boundary, where the subscript '$m$' denotes normal direction to the hole boundary (see Fig. 1), $\theta$ is an angle also shown in Fig. 1. $\mathbf{t}_m$ is the surface traction, $\Pi_1^0 = \{\sigma_{11}^0 \quad \sigma_{12}^0 \quad \sigma_{13}^0 \quad D_1^0\}^T$, $\Pi_2^0 = \{\sigma_{21}^0 \quad \sigma_{22}^0 \quad \sigma_{23}^0 \quad D_2^0\}^T$, and by using the coordinate transformation as well as applying (8), we have

$$h_m = -h_1 \sin\theta + h_2 \cos\theta = 2\tilde{k}\,\mathrm{Im}\,\{(\cos\theta + \tau \sin\theta) g''(z_t)\}. \tag{15}$$

Here 'Im' stands for the imaginary part, $\tilde{k} = \sqrt{k_{11}k_{22} - k_{12}^2}$. Since $\mathbf{t}_m$ is the surface traction at a point on the hole boundary, it can also be written as

$$\mathbf{t}_m = \partial\boldsymbol{\phi}/\partial n \tag{16}$$

where $n$ is the arc length measured along the hole boundary in the direction such that, when one faces the direction of increasing $n$, the material is located on the right-hand side (see Fig. 1). Thus, the boundary condition along the hole boundary, $(14)_2$, can be replaced equivalently by

$$\boldsymbol{\phi} = -x_1\Pi_2^0 + x_2\Pi_1^0 + \mathbf{K}_0, \quad \text{along the hole boundary} \tag{17}$$

where $\mathbf{K}_0$ represents a kind of rigid body motion. Since the rigid body motion is of no interest, we assume that $\mathbf{K}_0 = 0$.

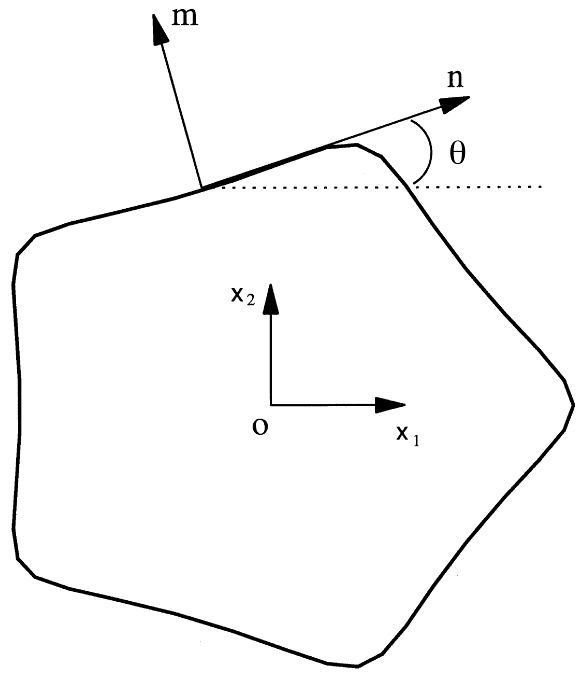


Fig. 1. Geometry of a particular hole ($a = 1, e = 1, k = 4, \eta = 0.1$).

### 3.2. Conformal mapping

In our analysis the contour of the hole is described by (Hwu, 1990a)

$$x_1 = a(\cos\psi + \eta\cos k\psi)$$
$$x_2 = a(e\sin\psi - \eta\sin k\psi) \tag{18}$$

where $0 < e \leqslant 1$, $k$ is an integer, and $\psi$ is a real parameter. By appropriate selection of the parameters $e$, $k$ and $\eta$, we can obtain various special kinds of holes, such as ellipse, circle, triangular, square and pentagon.

Recall from complex theory that if we have two complex domains in the $z$ and $\zeta$ planes, respectively, the conformal transformation is given by

$$z = w(\zeta) \tag{19}$$

where $w$ is a holomorphic function and $\zeta = re^{i\psi}$, where $r$ and $\psi$ are a pair of polar coordinates. To transform the exterior of a unit circle in the $\zeta$ plane onto the exterior of the hole in the $z$ plane, we use the following transformation (Hwu, 1990a):

$$z_\alpha = a(a_1\zeta + a_2\zeta^{-1} + a_3\zeta^k + a_4\zeta^{-k}) \tag{20}$$

in which

$$a_1 = (1 - ip_\alpha e)/2, \quad a_2 = (1 + ip_\alpha e)/2$$

$$a_3 = \eta(1+ip_\alpha e)/2, \quad a_4 = \eta(1-ip_\alpha e)/2 \tag{21}$$

The technique of conformal transformation tells us that $g(z_t)$ and $\mathbf{F}(\mathbf{Z})$ may be chosen as

$$g(z_t) = g^*(\zeta_t), \quad \mathbf{F}(\mathbf{Z}) = \mathbf{F}^*(\zeta) \tag{22}$$

where $g^*$ and $F^*$ can be expressed as some simple functions of $\zeta$.

### 3.3. Solution to temperature and electroelastic fields

We first study the solution to temperature $T$. Noting that along the hole boundary $x_1$ and $x_2$ are expressed by (18), and $\zeta = e^{i\psi}$, the boundary condition $(14)_1$ suggests that the arbitrary function $g(z_t)$ ought to be chosen in the form

$$g(z_t) = b_1\int\zeta^{-1}(z_t)\,\mathrm{d}z_t + b_k\int\zeta^{-k}(z_t)\,\mathrm{d}z_t \tag{23}$$

where $b_1$ and $b_k$ are two complex numbers to be determined. In using the boundary condition $(14)_1$, one needs to evaluate $g''(z_t)$ along the hole boundary. Knowing that $\zeta_t = e^{i\psi}$, and that $\theta$ (see Fig. 1) is related to $\psi$ by

$$a(\sin\psi + k\eta\sin k\psi) = \rho\cos\theta a(e\cos\psi - k\eta\cos k\psi) = -\rho\sin\theta \tag{24}$$

we have

$$\frac{\mathrm{d}\zeta_t}{\mathrm{d}z_t} = -\frac{i\,e^{i\psi}}{\rho(\cos\theta+\tau\sin\theta)}$$

$$g''(z_t) = \frac{i}{\rho(\cos\theta+\tau\sin\theta)}(b_1\,e^{-i\psi} + kb_k\,e^{-ik\psi}) \tag{25}$$

Substitution of $(25)_2$ into (15), and using $(14)_1$, we obtain

$$b_1 = -a(eh_1^0 + ih_2^0)/2\tilde{k}$$

$$b_k = a\eta(h_1^0 - ih_2^0)/2\tilde{k} \tag{26}$$

To obtain the explicit expression of $\boldsymbol{\phi}$, integrating (23) yields

$$g(z_t) = ab_1(a_{1\tau}\ln\zeta + a_{2\tau}\zeta^{-2}/2) \tag{27}$$

for $k = 1$,

$$g(z_t) = a(b_1a_{1\tau} + kb_ka_{3\tau})\ln\zeta + a(a_{2\tau}b_1 - a_{1\tau}b_k)\zeta^{-2}/2 + 3ab_1a_{3\tau}\zeta^2/2$$
$$+ a(3a_{4\tau}b_1 + a_{2\tau}b_k)\zeta^{-4}/4 + ab_ka_{4\tau}\zeta^{-6}/2 \tag{28}$$

for $k = 3$,

$$g(z_t) = a(b_1a_{1\tau} + kb_ka_{3\tau})\ln\zeta + aa_{2\tau}b_1\zeta^{-2}/2 + kab_1a_{3\tau}\zeta^{k-1}/(k-1) \tag{29}$$
$$+ a(ka_{4\tau}b_1 + a_{2\tau}b_k)\zeta^{-(k+1)}/(k+1) + a_{1\tau}b_k\zeta^{1-k}/(1-k) + ab_ka_{4\tau}\zeta^{-2k}/2$$

for other values of $k$, where $a_{i\tau}$ ($i = 1, 2\,3, 4$) are defined by

$$a_{1\tau} = (1 - i\tau e)/2, \quad a_{2\tau} = (1 + i\tau e)/2, \quad a_{3\tau} = \eta(1 + i\tau e)/2, \quad a_{4\tau} = \eta(1 - i\tau e)/2$$

By checking with (17) and (27) to (29), the general solution of $\mathbf{U}$ and $\boldsymbol{\phi}$ can now be assumed as

$$\mathbf{U} = 2\sum_{m=1}^{M} \mathrm{Re}\,\{\mathbf{AF}_m(\mathbf{Z})\mathbf{q}_m\} + 2\,\mathrm{Re}\,\{\mathbf{c}g(z_t)\}$$

$$\phi = 2\sum_{m=1}^{M} \mathrm{Re}\,\{\mathbf{BF}_m(\mathbf{Z})\mathbf{q}_m\} + 2\,\mathrm{Re}\,\{\mathbf{d}g(z_t)\} \tag{30}$$

where $M = 3$ for $k = 1$, $M = 7$ for $k = 3$, $M = 8$ for other values of $k$, and

$$\mathbf{F}_m(\mathbf{Z}) = \mathrm{diag}\,[f_m(z_1) \quad f_m(z_2) \quad f_m(z_3) \quad f_m(z_4)], \quad m = 1\text{–}8 \tag{31}$$

in which

$$f_1(z_j) = \zeta_j^{-1}, \quad f_2(z_j) = (1 + iep_j)\ln\zeta_j, \quad f_3(z_j) = (1 + iep_j)\zeta_j^{-2},$$
$$f_4(z_j) = \zeta_j^{-k}, \quad f_5(z_j) = (1 + iep_j)\zeta_j^{-(k+1)}, \quad f_6(z_j) = (1 + iep_j)\zeta_j^{-2k},$$
$$f_7(z_j) = (1 + iep_j)\zeta_j^{k-1}, \quad f_8(z_j) = (1 + iep_j)\zeta_j^{1-k}, \tag{32}$$

In order to express the right-sides of (30) in terms of real quantities, the arbitrary complex constant vector is replaced by

$$\mathbf{q}_m = \mathbf{A}^T\mathbf{q}_{ma} + \mathbf{B}^T\mathbf{q}_{mb} \tag{33}$$

where $\mathbf{q}_{ma}$ and $\mathbf{q}_{mb}$ are real constant vectors. To determine the unknown constants $\mathbf{q}_{ma}$ and $\mathbf{q}_{mb}$, we employ (17), (30) and the identities (11) and (12). Substituting (27)–(29) and (31)–(33) into $(14)_2$, we obtain

$$\mathbf{q}_{1a} = -a\Pi_2^0, \quad \mathbf{Lq}_{1b} - \mathbf{S}^T\mathbf{q}_{1a} = ae\Pi_1^0\mathbf{q}_{4a} = -a\eta\Pi_2^0, \quad \mathbf{Lq}_{4b} - \mathbf{S}^T\mathbf{q}_{4a} = -a\eta\Pi_1^0$$
$$(\mathbf{S}^T - e\mathbf{N}_1^T)\mathbf{q}_{2a} - (\mathbf{L} + e\mathbf{N}_3)\mathbf{q}_{2b} = 2a\,\mathrm{Im}\,\{(a_1b_1 + ka_3b_k)\mathbf{d}\}$$
$$\mathbf{O}_{11}\mathbf{q}_{ma} + \mathbf{O}_{12}\mathbf{q}_{mb} = -2\,\mathrm{Re}\,\{d_m\mathbf{d}\}\mathbf{O}_{21}\mathbf{q}_{ma} + \mathbf{O}_{22}\mathbf{q}_{mb} = -2\,\mathrm{Im}\,\{d_m\mathbf{d}\} \tag{34}$$

where $m = 3, 5, 6, 7, 8$, and

$$\mathbf{O}_{11} = \mathbf{I} + e(\mathbf{N}_1\mathbf{S}^T + \mathbf{N}_3\mathbf{H}), \quad \mathbf{O}_{12} = \mathbf{N}_3\mathbf{S} - \mathbf{N}_1^T\mathbf{L}$$
$$\mathbf{O}_{21} = \mathbf{N}_1^T - \mathbf{S}^T, \quad \mathbf{O}_{22} = \mathbf{L} + e\mathbf{N}_3$$
$$d_3 = aa_2b_1/2, \quad d_5 = a(a_2b_k + ka_4b_1)/(k+1), \quad d_6 = aa_4b_k/2,$$
$$d_7 = kaa_3b_1/(k-1), \quad d_8 = aa_1b_k/(1-k)$$

Equation (34) includes independent unknown vectors $\mathbf{q}_{ma}$ and $\mathbf{q}_{mb}$ ($m = 1$–$8$), so we need another equation to ensure the solution to be unique. Noting the multivalued properties of logarithmic functions, the equation can be obtained from the requirement that $\mathbf{U}$ is single-valued. Substituting the expression of $f_2(z_i)$ into $(30)_1$, the single-valued condition of displacement and electric potential requires that

$$(\mathbf{H}-e\mathbf{N}_2)\mathbf{q}_{2a}+(\mathbf{S}-e\mathbf{N}_1)\mathbf{q}_{2b} = 2a\,\mathrm{Im}\,\{(a_1b_1+ka_3b_k)\mathbf{c}\} \tag{35}$$

Combining (34) and (35), we obtain

$$\begin{aligned}
&\mathbf{q}_{1a} = -a\Pi_2^0, \quad \mathbf{q}_{1b} = a\mathbf{L}^{-1}(e\Pi_1^0-\mathbf{S}^T\Pi_2^0) \\
&\mathbf{q}_{4a} = -a\eta\Pi_2^0, \quad \mathbf{q}_{4b} = -a\eta\mathbf{L}^{-1}(\Pi_1^0+\mathbf{S}^T\Pi_2^0) \\
&\mathbf{q}_{2a} = 2a[(\mathbf{L}+e\mathbf{N}_3)^{-1}(\mathbf{S}^T-e\mathbf{N}_1^T)+(\mathbf{S}-e\mathbf{N}_1)^{-1}(\mathbf{H}-e\mathbf{N}_2)]^{-1}\{(\mathbf{S}-e\mathbf{N}_1)^{-1} \\
&\qquad \times \mathrm{Im}\,[(a_1b_1+ka_3b_k)\mathbf{c}]+(\mathbf{L}+e\mathbf{N}_3)^{-1}\,\mathrm{Im}\,[(a_1b_1+ka_3b_k)\mathbf{d}]\} \\
&\mathbf{q}_{2b} = 2a[(\mathbf{S}^T-e\mathbf{N}_1^T)^{-1}(\mathbf{L}+e\mathbf{N}_3)+(\mathbf{H}-e\mathbf{N}_2)^{-1}(\mathbf{S}-e\mathbf{N}_1)]^{-1}\{(\mathbf{H}-e\mathbf{N}_2)^{-1} \\
&\qquad \times \mathrm{Im}\,[(a_1b_1+ka_3b_k)\mathbf{c}]-(\mathbf{S}^T-e\mathbf{N}_1^T)^{-1}\,\mathrm{Im}\,[(a_1b_1+ka_3b_k)\mathbf{d}]\} \\
&\mathbf{q}_{mb} = 2(\mathbf{O}_{11}^{-1}\mathbf{O}_{12}-\mathbf{O}_{21}^{-1}\mathbf{O}_{22})^{-1}\{\mathbf{O}_{21}^{-1}\,\mathrm{Im}(d_m\mathbf{d})-\mathbf{O}_{11}^{-1}\,\mathrm{Re}(d_m\mathbf{d})\} \\
&\mathbf{q}_{ma} = 2(\mathbf{O}_{12}^{-1}\mathbf{O}_{11}-\mathbf{O}_{22}^{-1}\mathbf{O}_{21})^{-1}\{\mathbf{O}_{22}^{-1}\,\mathrm{Im}(d_m\mathbf{d})-\mathbf{O}_{12}^{-1}\,\mathrm{Re}(d_m\mathbf{d})\}
\end{aligned} \tag{36}$$

The stress and electric displacement can be obtained by using $(8)_2$, $(30)_2$ and (36). If $\mathbf{t}_n$ is the surface traction at a point along the hole boundary of which the normal is $\mathbf{n}$, we have

$$\mathbf{t}_n = -\boldsymbol{\phi}_{,m} = \boldsymbol{\phi}_{,1}\sin\theta-\boldsymbol{\phi}_{,2}\cos\theta \tag{37}$$

and then the normal stress $\sigma_{nn}$, the shear stresses $\sigma_{nm}$, $\sigma_{n3}$, as well as the electric displacement $D_n$ are, respectively, expressed as

$$\sigma_{nn} = \mathbf{n}^T(\theta)\mathbf{t}_n, \quad \sigma_{nm} = \mathbf{m}^T(\theta)\mathbf{t}_n, \quad \sigma_{n3} = (\mathbf{t}_n)_3, \quad D_n = (\mathbf{t}_n)_4 \tag{38}$$

## 4. Field intensity factors and energy release rate

### 4.1. Field intensity factors

As an application of the above solutions, we present here the field intensity factors and energy release rate for cracks. A crack of length $2a$ may be formed by letting $e$ and $\eta$ in (18) approach zero. The solution of SED for the crack problem can then be obtained from (8), (13) and $(30)_2$ by letting $e = \eta = 0$. Thus, the asymptotic form of SED, $\Pi_2$, ahead of the crack tip along the $x_1$-axis, can be given by

$$\Pi_2 = \Pi_*\frac{1}{\sqrt{x_1^2-a^2}} \quad (|x_1| > a) \tag{39}$$

where

$$\Pi_* = a\Pi_2^0+2\,\mathrm{Re}\,\{\mathbf{BA}^T(\mathbf{q}_{2a}-2\mathbf{q}_{3a})+\mathbf{BB}^T(\mathbf{q}_{2b}-2\mathbf{q}_{3b})\}$$

With the usual definition, the field intensity factors are given by

$$\mathbf{K} = \lim_{x_1\to a}\sqrt{2\pi(x_1-a)}\Pi_2 = \Pi_*\sqrt{\frac{\pi}{a}} \tag{40}$$

where $\mathbf{K} = \{K_{\text{II}}, K_{\text{I}}, K_{\text{III}}, K_D\}^T$, in which $K_{\text{I}}$, $K_{\text{II}}$, $K_{\text{III}}$ are the usual stress intensity factors, and $K_D$ is the so-called 'elastic displacement intensity factor'.

*4.2. Energy release rate*

The energy release rate can be obtained by considering the work done in closing the crack tip over an infinitesimal distance $\Delta x$, which can be calculated by (Pak, 1990)

$$G = \lim_{\Delta x_1 \to 0} \frac{1}{2\Delta x} \int_0^{\Delta x} \Pi_2^T(x) \Delta \mathbf{U}(x - \Delta x)\, \mathrm{d}x \tag{41}$$

where $G$ denotes the energy release rate. Note that the integration variable $x$ represents the distance ahead of the crack tip. Using the solution obtained previously, the jumps of elastic displacements and the electric potential (EDEP), $\Delta\mathbf{U}$, across the crack faces can be given by

$$\begin{aligned} \Delta\mathbf{U} &= \mathbf{U}(x_1, 0^+) - \mathbf{U}(x_1, 0^-) = i(\overline{\mathbf{AB}}^{-1} - \mathbf{AB}^{-1})\sqrt{a^2 - x_1^2}\,\Pi_2^0 \\ &+ i[(\mathbf{AA}^T - \overline{\mathbf{AA}}^T)\mathbf{q}_{3a} + (\mathbf{AB}^T - \overline{\mathbf{AB}}^T)\mathbf{q}_{3b}]\frac{4x_1}{a}\sqrt{a^2 - x_1^2} \\ &+ i(\mathbf{c}b_1 - \bar{\mathbf{c}}\bar{b}_1)\frac{x_1}{a}\sqrt{a^2 - x_1^2} \quad (0 < x_1 < a) \end{aligned} \tag{42}$$

By the substitution of (39) and (42) in (41), we have

$$G = \frac{i\pi}{2}\Pi^T{}_*\{(\overline{\mathbf{AB}}^{-1} - \mathbf{AB}^{-1})\Pi_2^0 + 4(\mathbf{AA}^T - \overline{\mathbf{AA}}^T)\mathbf{q}_{3a} + 4(\mathbf{AB}^T - \overline{\mathbf{AB}}^T)\mathbf{q}_{3b} + (\mathbf{c}b_1 - \bar{\mathbf{c}}\bar{b}_1)\} \tag{43}$$

## 5. Numerical illustration

Since the main purpose of this paper is to present the basic formulations of the proposed method and demonstrate its feasibility, the obtained results will be limited to an elliptic hole embedded in a square piezoelectric plate subjected to a uniform heat flow $h_2^0$ at the boundary $x_2 = \pm L$ (see Fig. 2). Thus, the boundary conditions are as follows:

$$\begin{aligned} &h_2 = h_2^0, \quad \Pi_2 = 0, \quad \text{on } x_2 = \pm L \\ &h_1 = \Pi_1 = 0, \quad \text{on } x_1 = \pm L \end{aligned} \tag{44}$$

In the analysis, we assume that $e = b/a = 0.5$, $L = 10a$ (see Fig. 2 for the geometrical meaning of $a$, $b$ and $L$). Since the value of $L$ is much larger than that of $a$, the present problem can be approximately viewed as an elliptic hole embedded in an infinite plate. For convenience we consider a piezoelectric ceramic ($BaTiO_3$) plate with an elliptic hole. The material constants of the plate are as follows (Dunn, 1993):

$$c_{11} = 150\ \text{GPa}, \quad c_{12} = 66\ \text{GPa}, \quad c_{22} = 146\ \text{GPa}, \quad c_{33} = 44\ \text{GPa},$$

$$\alpha_{11} = 8.53 \times 10^{-6}/\text{K}, \quad \alpha_{22} = 1.99 \times 10^{-6}/\text{K}, \quad \lambda_2 = 0.133 \times 10^5\ \text{N/CK},$$

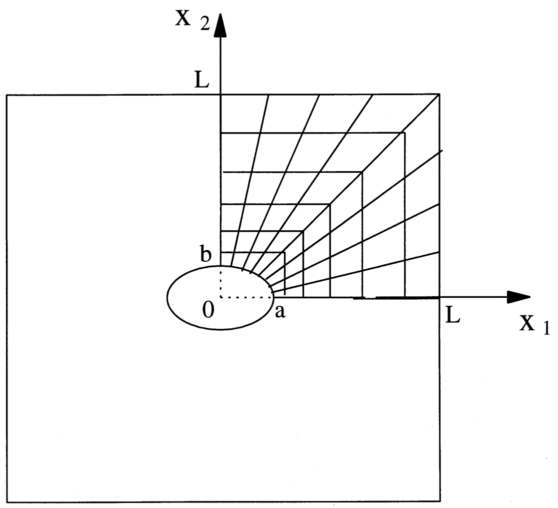


Fig. 2. Element mesh used in the example.

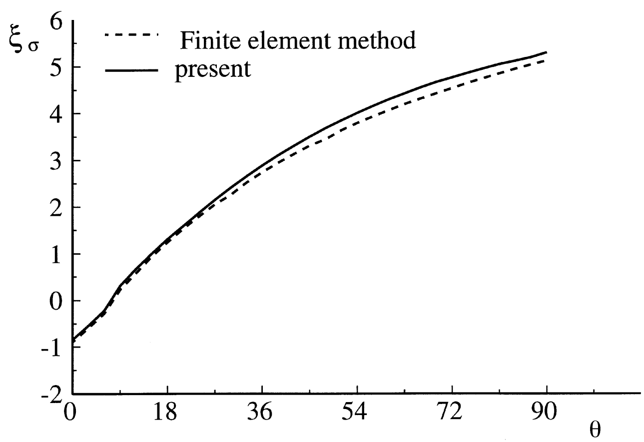


Fig. 3. Concentration parameter $\xi_\sigma$ vs angle $\theta$.

$$e_{21} = -4.35\ \text{C/m}^2, \quad e_{22} = 17.5\ \text{C/m}^2, \quad e_{13} = 11.4\ \text{C/m}^2, \quad \kappa_{11} = 1115\kappa_0,$$

$$\kappa_{22} = 1260\kappa_0, \quad \kappa_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{Nm}^2$$

Since the values of the coefficients of heat conduction for $BaTiO_3$ could not be found in the literature, the value $k_{22}/k_{11} = 1.5$ and $k_{12} = 0$ are assumed. We will calculate the concentration coefficients of SED, $\xi_\sigma = \tilde{k}\sigma_{nn}/a\gamma_{22}h_2^0$ and $\xi_D = \tilde{k}D_n/ag_2h_2^0$, along the hole boundary. However, the numerical results for such a problem are not available in the literature, to our knowledge. For comparison, the well-known finite element method is used to obtain the corresponding results. Owing to symmetry of the problem only one quarter of the problem is modelled by the element mesh shown in Fig. 2. Figures 3 and 4 show the results of $\xi_\sigma$ and $\xi_D$ vs the angle $\theta$ when $e = 0.5$,

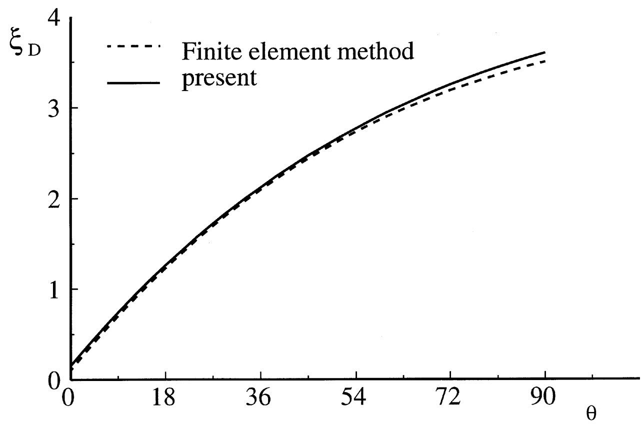


Fig. 4. Concentration parameter $\xi_D$ vs angle $\theta$.

and comparison is made with those obtained from finite element method. As is evident in Figs 3 and 4, both $\xi_\sigma$ and $\xi_D$ increase with the increase of angle $\theta$, and reach their maximum values at $\theta = 90°$.

## 6. Conclusions

The two-dimensional problem of a thermopiezoelectric sheet containing a hole of various shapes is studied. A unified analytical solution for the hole problem is derived through use of the extended Stroh formalism, conformal mapping and an ingenious selection of arbitrary function $\mathbf{F}(\mathbf{Z})$. The solution is suitable for analysing a wide range of hole problems, in which the hole may be an ellipse, a triangle, a square, a pentagon, and the like. As an application of the solution, the crack open displacements, field intensity factors, and energy release rate are derived. The numerical results obtained here are in good agreement with those obtained from the finite element method.

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